Controllability
Monday, June 22nd 2015

Solution Problem 1
Setting $x_1 = x,\ x_2 = y,\ x_3 = \dot{x}$ and $x_4 = \dot{y}$, the system can be written

$$\begin{aligned}\dot{x}_1(t) &= x_3(t)\\ \dot{x}_2(t) &= x_4(t)\\ \dot{x}_3(t) &= \alpha\cos(u(t))\\ \dot{x}_4(t) &= \alpha\sin(u(t))\end{aligned}$$

We want to minimize the Mayer-Lagrange cost $\int_0^{t_f} 1dt + g(x(t_f))$ There is no constraint on the control so we can apply the weak Pontryagin's Maximum principle. If u is optimal on $[0,t_f]$, there exists a continuous function $p : [0,t_f] \to \mathbb{R}^4 \setminus \{0\}$ and a real number $p^0 \le 0$ such that the pair $(p(.), p^0)$ is non trivial and, almost everywhere on $[0,t_f]$, we have

$$\dot{x}(t) = \frac{\partial H}{\partial p}(x(t),p(t),u(t)),\ \dot{p}(t) = -\frac{\partial H}{\partial x}(x(t),p(t),u(t))$$

and

$$\frac{\partial H}{\partial u}(x(t),p(t),u(t)) = 0$$

where

$$H(x,p,u) = p^0 + p_1x_3 + p_2x_4 + \alpha\big(p_3\cos(u) + p_4\sin(u)\big).$$

Therefore, $\dot{p}_1 = \dot{p}_2 = 0$ so the functions p_1 and p_2 are constant on $[0,t_f]$. Moreover, $\dot{p}_3 = p_1$ and $\dot{p}_4 = p_2$ so $p_3(t) = p_1t + p_3(0)$ and $p_4(t) = p_2t + p_4(0)$ on $[0,t_f]$. Since $\frac{\partial H}{\partial u}(x,p,u) = \alpha\big(-p_3\sin(u) + p_4\cos(u)\big)$, we find

$$\tan(u) = \frac{p_4}{p_3} = \frac{p_2t + p_4(0)}{p_3t + p_3(0)}.$$

Solution Problem 2
There is no constraint on the control so the weak Pontryagin's Maximum principle applies. The Hamiltonian function of the problem is

$$H(x,p,u) = p^0|u|^2 + \sum_{i=1}^{m} u_i\langle p, F_i(x)\rangle.$$

From the condition $\frac{\partial H}{\partial u}(x,p,u) = 0$, we get $2p^0 u_i + \langle p, F_i(x)\rangle = 0$, for all $1 \le i \le m$. We consider the normal case $p^0 < 0$ so, by homogeneity, we can set $p^0 = -\frac{1}{2}$. Thus, we get $u_i = \langle p, F_i(x)\rangle$, for all $0 \le i \le m$. Denoting $H_i(x,p) = \langle p, F_i(x)\rangle$ and plugging in H, we find that the extremal of the problem are the solutions to the true Hamiltonian system

$$\dot{x}(t) = \frac{\partial H_r}{\partial p}(x(t),p(t)),\ \dot{p}(t) = -\frac{\partial H_r}{\partial x}(x(t),p(t))$$

where $H_r(x,p) = H_0(x,p) + \frac{1}{2}\sum_{i=0}^{m} H_i^2(x,p)$.

Solution Problem 3

1. Reparametrizing, the L^2-cost that we want to minimize is

$$\int_0^{l_f} |u|^2 \frac{dl}{\omega(l,x)}.$$

   Since there is no constraint on the control and we consider the normal case, the weak maximum princile applies. Setting $p^0 = -\frac{1}{2}$ and using a similar reasoning as in the solution of the problem 2, we show that the optimal control is given by

$$u(l,z) = \omega(l,x)(H_1,\ldots,H_m)(l,z)$$

   and that z is an integral curve of the Hamiltonian

$$H_n(l,z) = \frac{\omega(l,x)}{2}\sum_{i=1}^{m} H_i^2(l,z).$$

2. Since $H_n(l,x,\cdot)$ is quadratic in the adjoint state p, i.e $H_n(l,x,\alpha p) = \alpha^2 H_n(l,x,p)$, we get

$$\frac{d\tilde{x}}{ds} = \frac{\partial H_n}{\partial p}(\frac{s}{\epsilon},\tilde{x},\tilde{p}),\ \frac{d\tilde{p}}{ds} = -\frac{\partial H_n}{\partial x}(\frac{s}{\epsilon},\tilde{x},\tilde{p}).$$

3. We can assume that $\omega(l,x) = 1$. Calculating, we find that

$$H_n(l,x,p) = \frac{p^2}{2}\cos^2(l)$$

   Therefore, the function p is constant.The averaged Hamiltonian is given by

$$\begin{aligned} H(x,p) &= \frac{1}{2\pi}\int_0^{2\pi} H_n(l,z)dl \\ &= \frac{p^2}{4\pi}\int_0^{2\pi}\cos^2(l)dl \\ &= \frac{p^2}{4}. \end{aligned}$$

Integrating, we find that

$$\frac{dx}{dl} = \frac{\partial H}{\partial p}(l,x,p) = \frac{p}{2}$$

so

$$x(l) = \frac{p}{2}l + x(0).$$

On the other hand, substituting l by $\frac{s}{\epsilon}$ in the expression of H_n, we get

$$\frac{d\tilde{x}}{ds} = \tilde{p}\cos^2\big(\frac{s}{\epsilon}\big),\ \ \frac{d\tilde{p}}{ds} = 0.$$

Integrating, we find that $\tilde{p}$ is constant and

$$\tilde{x}(s) = \tilde{p}\int cos^2\big(\frac{s}{\epsilon}\big)ds = \frac{\tilde{p}}{2}\Big(s + \frac{\epsilon}{2}\sin\big(\frac{2s}{\epsilon}\big)\Big) + \tilde{x}(0).$$

The function sin being bounded on $\mathbb{R}$, it is clear that, for given initial conditions, the pair $(\tilde{x},\tilde{p})$ converges uniformly on $[0,1]$ towards the pair (x,p).